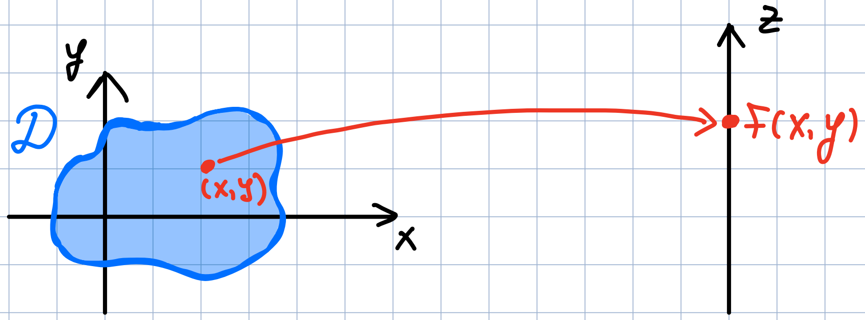
Functions of Several Variables and Level Curves

REVIEW

Function of two variables:

$$f : (x,y) \longmapsto f(x,y)$$

in $\underbrace{\mathcal{D}}_{\text{domain}} \subset \mathbb{R}^2$ in $\mathbb{R}$

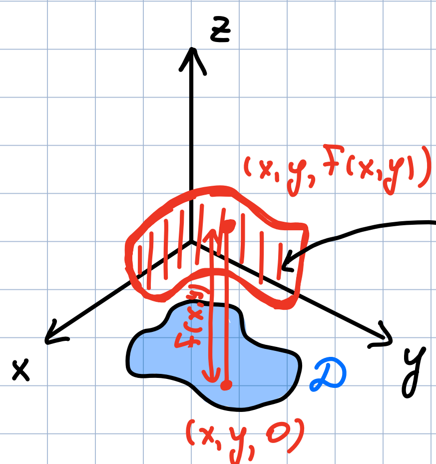


range: set of all values of $f(x,y) \subset \mathbb{R}$

domain: all (x,y) s.t. $f(x,y)$ is well-defined.

Graph: Set of points (x,y,z) in $\mathbb{R}^3$

with $(x,y) \in \mathcal{D}$, $z = f(x,y)$



graph (surface)
sitting above $\mathcal{D}$

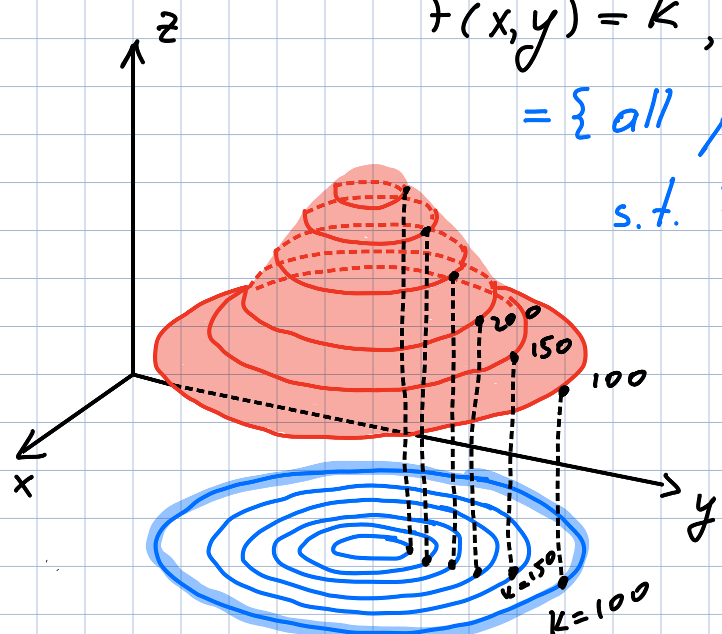
Functions of 3 variables:

$$f(x,y,z)$$

Level curves: curves with eq.

$f(x,y) = k$, i.e.

$= \{$ all points (x,y)
s.t. $f(x,y) = k \}$



on this curve $f(x,y) = 100$

Problems

① Find and sketch the domain of the function $f(x,y) = \ln(1+x+y)$

② Sketch the graph of the function $f(x,y) = 1-y^2$

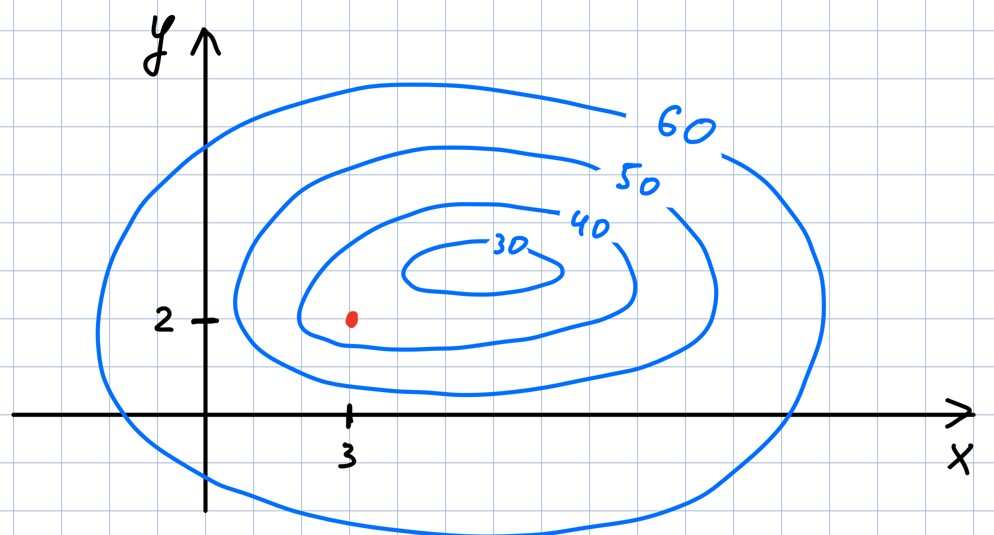
③ Evaluate the limit or show that it doesn't exist:

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{2x^2+y^2}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy+1}{2x^2+y^2+3}$

c) $\lim\limits_{(x,y)\to(1,1)} \dfrac{2xy}{2x^2+y^2}$

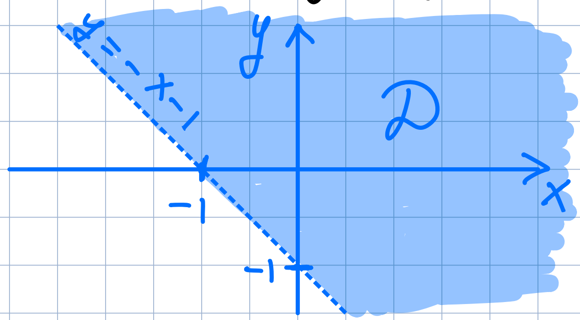
④ Estimate the value $f(3,2)$



Solutions

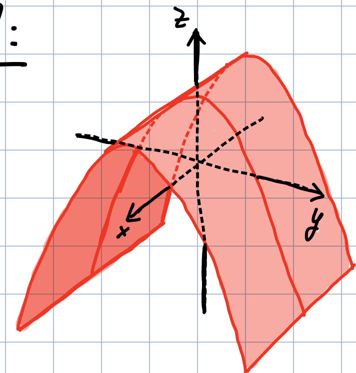
① Find and sketch the domain of the function $f(x,y) = \ln(1+x+y)$

Sol: $D = \{ (x,y) \mid y > -1-x \}$

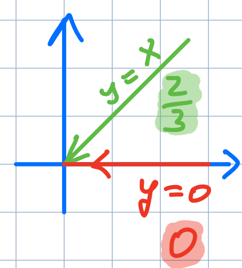


② Sketch the graph of the function $f(x,y) = 1-y^2$

Sol:



③ Evaluate the limit or show that it doesn't exist:

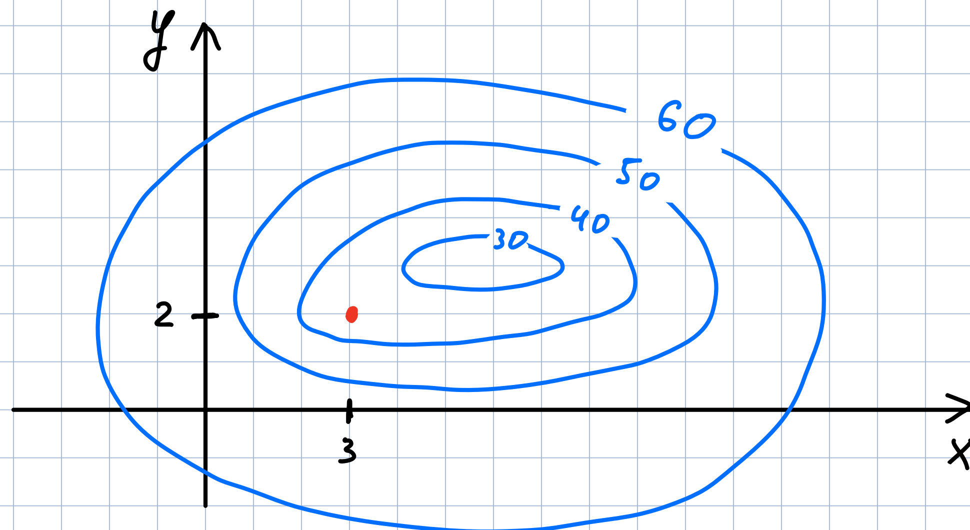


a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{2x^2+y^2}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy+1}{2x^2+y^2+3} = \dfrac{1}{3}$

c) $\lim\limits_{(x,y)\to(1,1)} \dfrac{2xy}{2x^2+y^2} = \dfrac{2}{3}$

④ Estimate the value $f(3,2)$



Sol: $f(3,2) \approx 37$

PARTIAL DERIVATIVES

Partial derivatives

REVIEW

$f(x,y)$ partial derivative w.r.t. x at (a,b):

set $g(x) = f(\underbrace{x}_{\text{variable}}, b)$ ← constant

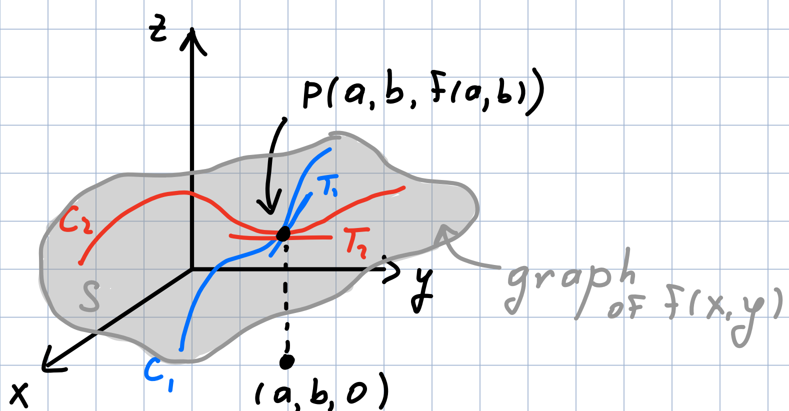
↑ function of single variable

$$\boxed{f_x(a,b) = g'(a)}$$

I.e. to find $f_x(x,y)$, view y as a constant, differentiate w.r.t. x

Notation: $\frac{\partial f}{\partial x} = f_x(x,y)$

Interpretations of partial derivatives:



C_1: graph of $g(x) = f(x,b)$

= intersection of S with plane $y=b$

C_2: graph of $h(y) = f(a,y)$

= intersection of S with plane $x=a$

$f_x(a,b)$ = slope of tangent line T_1 to C_1 at $P(a,b,f(a,b))$

$f_y(a,b)$ = slope of tangent line T_2 to C_2 at $P(a,b,f(a,b))$

Problems

① $u = x + yz$, $x = t$, $y = \ln t$, $z = \sin t$. Find $\frac{du}{dt}$.

② $z = z(x,y)$ defined by $x^2 + y^5 + z^3 = 0$ (*). Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\boxed{\text{Solutions}}$$

① $u = x + yz$, $x = t$, $y = \ln t$, $z = \sin t$. Find $\frac{du}{dt}$.

Sol: $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = 1 \cdot 1 + z\frac{1}{t} + y\cos t$

$= 1 + \frac{\sin t}{t} + \ln t \cos t$

---

② $z = z(x,y)$ defined by $x^2 + y^5 + z^3 = 0$ $(*)$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Sol:

$\frac{\partial}{\partial x}(*)$: $2x + 0 + 3z^2\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2x}{3z^2}$

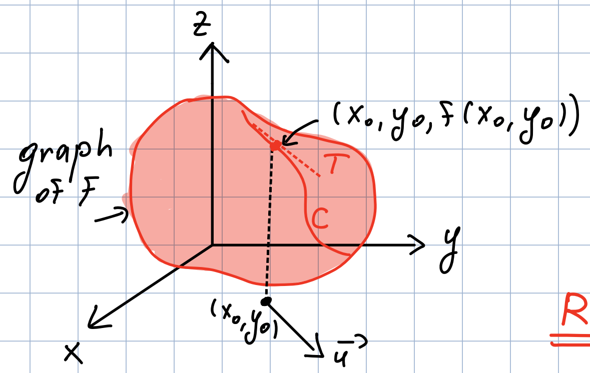
$\frac{\partial}{\partial y}(*)$: $0 + 5y^4 + 3z^2\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{5y^4}{3z^2}$

Directional derivatives and gradient

REVIEW

$f(x,y)$ $\quad$ $\vec{u} = \langle a, b \rangle$ — unit vector

$$\underbrace{D_{\vec{u}} f(x_0,y_0)}_{\text{directional derivative of } f \text{ at } (x_0,y_0) \text{ in the direction of } \vec{u}} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0,y_0)}{h}$$



$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b}$$

Rmk: $\vec{u} = \langle 1, 0 \rangle \Rightarrow D_{\vec{u}} f = f_x$; $\vec{u} = \langle 0, 1 \rangle \Rightarrow D_{\vec{u}} f = f_y$

Gradient: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ $\qquad$ $D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}$

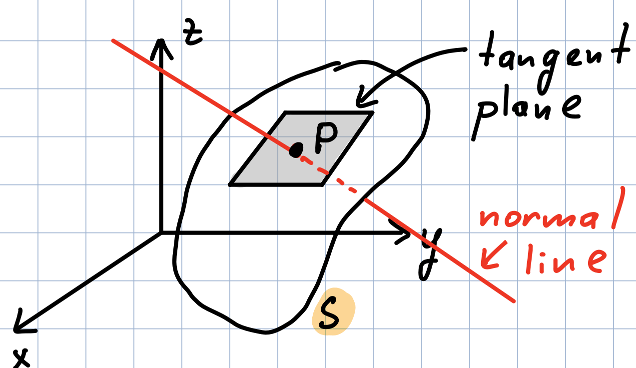
S: $F(x,y,z) = k$ — level surface

$P(x_0, y_0, z_0)$ point on S

The gradient $\nabla F(x_0,y_0,z_0)$ is orthogonal to the level surface S

the tangent plane to level surface $F(x,y,z) = k$ at $P(x_0,y_0,z_0)$:

$$\boxed{F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0}$$



normal line to S at P:

line through P and orthogonal to tangent plane (i.e. it's direction vector is $\nabla F(x_0,y_0,z_0)$)

$$\Rightarrow \frac{x - x_0}{F_x(x_0,y_0,z_0)} = \frac{y - y_0}{F_y(x_0,y_0,z_0)} = \frac{z - z_0}{F_z(x_0,y_0,z_0)}$$

$\nabla F(x_0,y_0,z_0)$ - direction of maximal increase of F

True of False?

The gradient vector $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x,y) = f(x_0, y_0)$ at the point (x_0, y_0).

Q: In which direction does $f(x,y)$ (or $f(x,y,z)$) change fastest? what is maximum rate of change?

Problems

① Find eqs. of the tangent plane and normal line to the surface $z = e^x \cos y$ at the point $(0,0,1)$

② Find the directional de rivative tion of $f(x,y) = x^2 e^{-y}$ in the direc $\langle 2, -3 \rangle$ at the point $(-2, 0)$

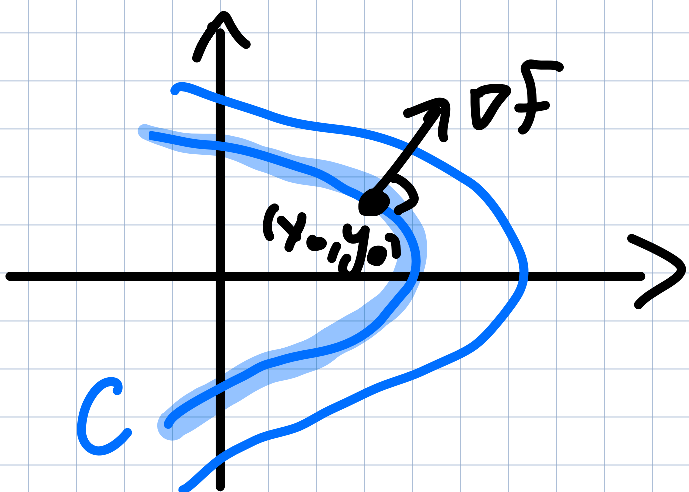
③ Find the direction in which $f(x,y,z) = ze^{xy}$ increases most rapidly at pt $(0,1,2)$ what is the max rate of increase?

$\boxed{\text{Solutions}}$

True of False?

The gradient vector $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x,y) = f(x_0, y_0)$ at the point (x_0, y_0).

TRUE:



What is a level curve?

A level curve $f(x,y) = a$ is the set of all points (x,y) where $f(x,y) = a$.

A tangent vector to the level curve:

Let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be a parametriz. of the level curve $f(x,y) = a$.

Then $f(x(t), y(t)) = a$ and

$$\frac{d}{dt} f(x(t), y(t)) = \frac{d}{dt} a = 0$$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) = \nabla f(x,y) \cdot \underbrace{\langle x'(t), y'(t) \rangle}_{\vec{r}\,'(t)}$$

$\Rightarrow \nabla f \cdot \vec{r}\,'(t) = 0 \Rightarrow \nabla f \perp \vec{r}\,'(t)$. Recall that $\vec{r}\,'(t)$ is a tangent vector.

Q: In which direction does $f(x,y)$ (or $f(x,y,z)$) change fastest?
what is maximum rate of change?

A: Maximum value of directional derivative is $|\nabla f|$,
it occurs when $\vec{u}$ has same direction as $\nabla f(x,y)$

---

① Find eqs. of the tangent plane and normal line to the surface $z = e^x \cos y$ at the point $(0,0,1)$

Sol: $F(x,y,z) = e^x \cos y - z = 0$

$\nabla F = \langle e^x \cos y, -e^x \sin y, -1 \rangle$

$\nabla F(0,0,1) = \langle 1, 0, -1 \rangle$

tangent plane: $x - (z-1) = 0.$

normal line: $\frac{x}{1} = \frac{z-1}{-1}$, $y = 0.$

② Find the directional derivative of $f(x,y) = x^2 e^{-y}$ in the direction $\langle 2,-3 \rangle$ at the point $(-2,0)$

Sol: $u = \frac{\langle 2,-3 \rangle}{\sqrt{4+9}} = \langle \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \rangle$

$\nabla f = \langle 2xe^{-y}, -x^2 e^{-y} \rangle$

$\nabla f(-2,0) = \langle -4, -4 \rangle$

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = -\frac{4 \cdot 2}{\sqrt{13}} + \frac{4 \cdot 3}{\sqrt{13}} = \frac{4}{\sqrt{13}}$

---

③ Find the direction in which $f(x,y,z) = ze^{xy}$ increases most rapidly at pt $(0,1,2)$
What is the max rate of increase?

Sol: $\nabla f(0,1,2) = \langle yze^{xy}, xze^{xy}, e^{xy} \rangle = \langle 2, 0, 1 \rangle$; $|\nabla f| = \sqrt{2^2+0^2+1^2} = \sqrt{5}.$

$u = x + yz$ <u>Find $\frac{du}{dt}$</u>, where $x(t)$, $y(t)$ and $z(t)$ are given by

1) $x = t$, $y = e^t$, $z = \sin t$

2) $x = t$, $y = \ln t$, $z = e^t$

3) $x = e^t$, $y = \ln t$, $z = \sin t$

4) $x = e^t$, $y = 2t$, $z = t^3$

5) $x = \ln t$, $y = 2t^2$, $z = t$

6) $x = t + t^2$, $y = e^t$, $z = \ln t$

$u = x + yz \Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = (1)\,x'(t) + (z)\,y'(t) + (y)\,z'(t)$

1) $x = t,\ y = e^t,\ z = \sin t \qquad x'(t) = 1,\ \ y'(t) = e^t \qquad z'(t) = \cos t$

$$\frac{du}{dt} = 1 + \sin t \cdot e^t + e^t \cdot \cos t$$

2) $x = t,\ y = \ln t,\ z = e^t \qquad x'(t) = 1,\ \ y'(t) = \frac{1}{t},\ \ z'(t) = e^t$

$$\frac{du}{dt} = 1 + \frac{e^t}{t} + \ln t \cdot e^t$$

3) $x = e^t,\ y = \ln t,\ z = \sin t \qquad x'(t) = e^t,\ \ y'(t) = \frac{1}{t},\ \ z'(t) = \cos t$

$$\frac{du}{dt} = e^t + \frac{\sin t}{t} + \ln t \cdot \cos t$$

4) $x = e^t,\ y = 2t,\ z = t^3 \qquad x'(t) = e^t,\ \ y'(t) = 2,\ \ z'(t) = 3t^2$

$$\frac{du}{dt} = e^t + 2t^3 + (2t)\,3t^2$$

5) $x = \ln t,\ y = 2t^2,\ z = t \qquad x'(t) = \frac{1}{t},\ \ y'(t) = 4t,\ \ z'(t) = 1$

$$\frac{du}{dt} = \frac{1}{t} + 4t^2 + 2t^2$$

6) $x = t + t^2,\ y = e^t,\ z = \ln t \qquad x'(t) = 1 + 2t \qquad y'(t) = e^t,\ \ z'(t) = \frac{1}{t}$

$$\frac{du}{dt} = (1 + 2t) + \ln t \cdot e^t + \frac{e^t}{t}$$